

Homework 4, Spring 2023:

Let  $\mathcal{M}(f)$  and  $M(f)$  be the standard centered and uncentered Hardy-Littlewood maximal functions.

Problem 4.1: Define the centered Hardy-Littlewood maximal function  $\mathcal{M}_c$  and the uncentered Hardy-Littlewood maximal function  $M_c$  using cubes with sides parallel to the axes instead of balls in  $\mathbb{R}^n$ . Prove that

$$v_n(n/2)^{n/2} \leq \frac{M(f)}{M_c(f)} \leq 2^n/v_n, \quad v_n(n/2)^{n/2} \leq \frac{\mathcal{M}(f)}{\mathcal{M}_c(f)} \leq 2^n/v_n,$$

where  $v_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Conclude that  $\mathcal{M}_c$  and  $M_c$  are weak type  $(1, 1)$  and they map  $L^p(\mathbb{R}^n)$  to itself for  $1 < p \leq \infty$ .

Problem 4.2: Prove that for any fixed  $1 < p < \infty$ , the operator norm of  $M$  on  $L^p(\mathbb{R}^n)$  tends to infinity as  $n \rightarrow \infty$ .