Homework 4:

Let $\mathcal{M}(f)$ and $M(f)$ be the standard centered and uncentered Hardy-Littlewood maximal functions.

Problem 4.1: Define the centered Hardy-Littlewood maximal function $\mathcal{M}_c$ and the uncentered Hardy-Littlewood maximal function M_c using cubes with sides parallel to the axes instead of balls in $\mathbb{R}^n$. Prove that

$$v_n(n/2)^{n/2} \leq \frac{M(f)}{M_c(f)} \leq 2^n/v_n, \quad v_n(n/2)^{n/2} \leq \frac{\mathcal{M}(f)}{\mathcal{M}_c(f)} \leq 2^n/v_n,$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Conclude that $\mathcal{M}_c$ and M_c are weak type $(1,1)$ and they map $L^p(\mathbb{R}^n)$ to itself for $1 < p \leq \infty$.

Problem 4.2: Prove that for any fixed $1 < p < \infty$, the operator norm of M on $L^p(\mathbb{R}^n)$ tends to infinity as $n \to \infty$.