THE NUMBER OF ZEROS OF CROSS-PRODUCT BESSEL FUNCTIONS

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Abstract. For any $\nu > 0$ we study the following cross-product combination of Bessel functions

$$\mathfrak{h}_\nu(z) = j'_\nu(Rz)y'_\nu(rz) - j'_\nu(rz)y'_\nu(Rz), \quad 0 < r < R.$$

We give the number of zeros of $\mathfrak{h}_\nu(z)$ within the circle $|z| = (s + 1/2)\pi/(R - r)$ for large integer s.

1. Introduction

Given $0 < r < R < \infty$ and $\nu > 0$ we study the number of zeros of the following cross-product combination of Bessel functions

$$\mathfrak{h}_\nu(z) = j'_\nu(Rz)y'_\nu(rz) - j'_\nu(rz)y'_\nu(Rz),$$

where

$$j_\nu(z) = z^{1-d/2}J_\nu(z)$$

and

$$y_\nu(z) = z^{1-d/2}Y_\nu(z)$$

with $d \geq 3$ and J_ν and Y_ν the Bessel functions of first and second kind of order ν. The function j_ν and y_ν are (ultra)spherical Bessel functions of the first and second kind when ν takes certain special value. Here j'_ν and y'_ν represent derivatives with respect to z.

Cross-products of Bessel functions arise regularly in a variety of physical and mathematical problems with circular or cylindrical geometry. Sometimes it is very helpful to know properties of their zeros. For instance, Cochran [2] investigated

$$\mathfrak{f}_\nu(z) = J_\nu(Rz)Y_\nu(rz) - J_\nu(rz)Y_\nu(Rz) \tag{1.1}$$

and

$$\mathfrak{g}_\nu(z) = J'_\nu(Rz)Y'_\nu(rz) - J'_\nu(rz)Y'_\nu(Rz), \tag{1.2}$$

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and showed that for any fixed $\nu > 0$ there are $2s$ and $2s+2$ zeros of $\mathfrak{f}_\nu(z)$ and $\mathfrak{g}_\nu(z)$ respectively within the circle $|z| = (s+1/2)\pi/(R-r)$ for large integer s. This information can be used in the study of Weyl's law for the Dirichlet/Neumann Laplacian associated with the planar annulus

$$\mathscr{D} = \left\{x \in \mathbb{R}^2 : r < |x| < R\right\}.$$

Indeed, in [5] the first author, Müller, Wang and Wang obtained an asymptotics of the eigenvalue counting function with an improved remainder estimate for the Dirichlet Laplacian associated with $\mathscr{D}$. The connection between the eigenvalue counting and zeros of cross-product Bessel functions lies in the fact that the eigenvalues under investigation correspond to squares of zeros of $\mathfrak{f}_\nu(z)$, which can be easily verified by using the standard separation of variables. Hence in order to count eigenvalues, one needs to study zeros. In particular, the number of zeros is needed in the process of determining asymptotics of zeros (see the proof of [5, Theorem 2.8]).

It is also well-known that nonzero eigenvalues of the Neumann Laplacian associated with $\mathscr{D}$ correspond to squares of zeros of $\mathfrak{g}_\nu(z)$ (namely, $\mathfrak{h}_\nu$ with $d=2$).

We would like to consider the Neumann Laplacian and generalize the planar annulus in [5] to spherical shells in high dimensions. Hence we need to study zeros of $\mathfrak{h}_\nu(z)$ with $d \geq 3$. As a first step, we prove the following result on the number of zeros of $\mathfrak{h}_\nu(z)$.

Theorem 1.1. *Let $\mathfrak{h}_\nu(z)$ be defined as above. Then there is a large constant $C > 0$ such that for any $\nu > 0$, if $s \in \mathbb{N}$ satisfies $s > C(\nu^3+1)$, then $\mathfrak{h}_\nu(z)$ has precisely $2s + 2\delta(\nu)$ zeros within the circle $|z| = (s+1/2)\pi/(R-r)$, where $\delta(\nu)$ is 0 if $\nu = d/2-1$ and 1 otherwise.*

Remark 1.2. We follow the convention that zeros are counted with multiplicities.

Remark 1.3. Our result provides an explicit dependence on ν of the size of s, which is implicitly in the work of Cochran [2]. This is achievable due to our effort in tracking values of coefficients of expansions shown up in the proof.

Remark 1.4. At last we would like to mention a few interesting results on other properties of zeros. Asymptotic expansions for real zeros of $\mathfrak{f}_\nu(z)$ and $\mathfrak{g}_\nu(z)$ were derived in McMahon [6] (and also in Truell [8] for first zeros of $\mathfrak{g}_\nu(z)$ and $\nu = 1, 2, 3, 4$). Cochran [4] showed zeros of both $\mathfrak{f}_\nu(z)$ and $\mathfrak{g}_\nu(z)$ are analytic. Cochran [3] obtained analogous results for nonnegative real ν-zeros of $\mathfrak{f}_\nu(z)$ and $\mathfrak{g}_\nu(z)$.

In this paper, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are the Hankel functions of the first and second kind of order ν (also known as the Bessel functions of the third kind). It is well known that

$$Y_\nu(z) = \frac{\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)}, \tag{1.3}$$

where the right hand side of this equation is replaced by its limiting value if ν is an integer,

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \tag{1.4}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \tag{1.5}$$

and

$$\mathscr{C}_\nu'(z) = \frac{\mathscr{C}_{\nu-1}(z) - \mathscr{C}_{\nu+1}(z)}{2}, \tag{1.6}$$

where $\mathscr{C}$ may denote J, Y, $H^{(1)}$ and $H^{(2)}$. See 9.1.2–9.1.4 and 9.1.27 in [1]. The Landau notation $f = O(g)$ means $|f| \leqslant Cg$ for some constant C.

2. Analyticity of cross-product Bessel functions

In this section we study the analyticity of the following function

$$h_\nu(z) = \mathfrak{g}_\nu(z) + \frac{\left(1-\frac{d}{2}\right)^2}{Rrz^2}\mathfrak{f}_\nu(z) + \frac{1-\frac{d}{2}}{Rrz}\left(r\mathfrak{l}_\nu(z) - R\tilde{\mathfrak{l}}_\nu(z)\right), \tag{2.1}$$

where

$$\mathfrak{l}_\nu(z) = J_\nu(Rz)Y_\nu'(rz) - J_\nu'(rz)Y_\nu(Rz) \tag{2.2}$$

and

$$\tilde{\mathfrak{l}}_\nu(z) = J_\nu(rz)Y_\nu'(Rz) - J_\nu'(Rz)Y_\nu(rz). \tag{2.3}$$

It is easy to verify that

$$\mathfrak{h}_\nu(z) = (Rr)^{1-\frac{d}{2}}z^{2-d}h_\nu(z). \tag{2.4}$$

The following result tells us that the origin is not a zero of $h_\nu(z)$.

Lemma 2.1. *Let $h_\nu(z)$ be defined by (2.1) and $0 < r < R$ given. For all $\nu > 0$, we have that*

(1) if $\nu = d/2 - 1$, $h_\nu(z)$ is an entire function and $h_\nu(0) \neq 0$;
(2) if $\nu \neq d/2 - 1$, $h_\nu(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a pole at 0 of the second order.

Proof. We are going to derive ascending series of $h_\nu(z)$ by using ascending series of the Bessel functions.

We first prove the case $\nu = n \in \mathbb{N} \setminus \{1\}$. From [1, P. 360] or [7, P. 57, P. 243], we have that for general real ν, if $\nu \neq -1, -2, \ldots$, then

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)} = \left(\frac{z}{2}\right)^\nu \left(\frac{1}{\Gamma(\nu+1)} - \frac{\left(\frac{z}{2}\right)^2}{\Gamma(\nu+2)} + \cdots\right), \tag{2.5}$$

and if $\nu = n \in \mathbb{N}$ then

$$\begin{aligned} Y_n(z) =& \frac{-1}{\pi}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k} + \frac{2}{\pi}\ln\left(\frac{z}{2}\right) J_n(z) \\ & - \frac{1}{\pi}\left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \left(\psi(k+1) + \psi(n+k+1)\right) \frac{(-1)^k \left(\frac{z}{2}\right)^{2k}}{k!(n+k)!} \\ =& \begin{cases} \left(\frac{z}{2}\right)^{-1}\left(\frac{-1}{\pi} - \frac{\psi(1)+\psi(2)}{\pi}\left(\frac{z}{2}\right)^2 + \cdots\right) + \frac{2}{\pi}\ln\left(\frac{z}{2}\right) J_1(z) & \text{if } n = 1, \\ \left(\frac{z}{2}\right)^{-n}\left(\frac{-\Gamma(n)}{\pi} - \frac{\Gamma(n-1)}{\pi}\left(\frac{z}{2}\right)^2 + \cdots\right) + \frac{2}{\pi}\ln\left(\frac{z}{2}\right) J_n(z) & \text{if } n \geq 2, \end{cases} \end{aligned} \tag{2.6}$$

where

$$\psi(1) = -\gamma, \quad \psi(k) = -\gamma + \sum_{l=1}^{k-1} \frac{1}{l}, \quad k \geq 2,$$

and γ is the Euler's constant. Here we choose any branch of the Bessel functions correspond to the range [1] $-\pi + 2m\pi < \arg z < \pi + 2m\pi$ with $m \in \mathbb{Z}$.

By (2.5) and (2.6) we have

$$\begin{aligned} \mathfrak{f}_n(z) =& \left(\frac{r^{2n} - R^{2n}}{n\pi r^n R^n} + \left(\frac{r^{2n-2} - R^{2n-2}}{n(n-1)\pi r^{n-2} R^{n-2}} \right.\right. \\ & \left.\left. - \frac{r^{2n+2} - R^{2n+2}}{(n+1)n\pi r^n R^n}\right)\left(\frac{z}{2}\right)^2 + \cdots\right) + \mathfrak{R}_{\mathfrak{f}_n}(z), \end{aligned} \tag{2.7}$$

where

$$\mathfrak{R}_{\mathfrak{f}_n}(z) = \frac{2}{\pi}\ln\left(\frac{r}{R}\right) J_n(rz) J_n(Rz).$$

[1] Here $\arg z$ denotes the argument of the complex number z.

Applying the recurrence relation (1.6), (2.5) and (2.6) to (1.2) yields

$$\begin{aligned}\mathfrak{g}_n(z) =&\Bigg(\frac{n\left(R^{2n}-r^{2n}\right)}{4\pi r^{n+1}R^{n+1}}\left(\frac{z}{2}\right)^{-2}+\frac{(n-2)\left(R^{2n-2}-r^{2n-2}\right)}{4(n-1)\pi r^{n-1}R^{n-1}} \\ &-\frac{(n+2)\left(R^{2n+2}-r^{2n+2}\right)}{4(n+1)\pi r^{n+1}R^{n+1}}+\cdots\Bigg)+\mathfrak{R}_{\mathfrak{g}_n}(z),\end{aligned} \tag{2.8}$$

where

$$\mathfrak{R}_{\mathfrak{g}_n}(z)=\frac{1}{2\pi}\ln\left(\frac{r}{R}\right)\left(J_{n-1}(rz)-J_{n+1}(rz)\right)\left(J_{n-1}(Rz)-J_{n+1}(Rz)\right).$$

By a similar argument we obtain

$$\mathfrak{l}_n(z)=\left(\frac{R^{2n}+r^{2n}}{2\pi r^{n+1}R^n}\left(\frac{z}{2}\right)^{-1}+\frac{C_{\mathfrak{l}_n,1}}{2}\frac{z}{2}+\cdots\right)+\mathfrak{R}_{\mathfrak{l}_n}(z) \tag{2.9}$$

and

$$\tilde{\mathfrak{l}}_n(z)=\left(\frac{r^{2n}+R^{2n}}{2\pi R^{n+1}r^n}\left(\frac{z}{2}\right)^{-1}+\frac{C_{\tilde{\mathfrak{l}}_n,1}}{2}\frac{z}{2}+\cdots\right)+\mathfrak{R}_{\tilde{\mathfrak{l}}_n}(z), \tag{2.10}$$

where

$$C_{\mathfrak{l}_n,1}=\frac{(n-2)R^n}{n(n-1)\pi r^{n-1}}-\frac{R^{n+2}}{(n+1)\pi r^{n+1}}+\frac{r^{n-1}}{(n-1)\pi R^{n-2}}-\frac{(n+2)r^{n+1}}{n(n+1)\pi R^n}$$

and

$$\mathfrak{R}_{\mathfrak{l}_n}(z)=\frac{1}{\pi}\ln\left(\frac{r}{R}\right)J_n(Rz)\left(J_{n-1}(rz)-J_{n+1}(rz)\right)$$

and the constant $C_{\tilde{\mathfrak{l}}_n,1}$ and the term $\mathfrak{R}_{\tilde{\mathfrak{l}}_n}(z)$ can be obtained from $C_{\mathfrak{l}_n,1}$ and $\mathfrak{R}_{\mathfrak{l}_n}(z)$ by swapping r and R respectively.[2]

Then applying (2.7)–(2.10) to all factors in $h_n(z)$ defined by (2.1) yields

$$h_n(z)=\left(\left(n^2-\left(1-\frac{d}{2}\right)^2\right)\frac{\left(R^{2n}-r^{2n}\right)}{n\pi r^{n+1}R^{n+1}}\frac{1}{z^2}+C_{h_n,0}+\cdots\right)+\mathfrak{R}_{h_n}(z), \tag{2.11}$$

where

$$\begin{aligned}C_{h_n,0}=&\frac{\left((n-2)n-\left(1-\frac{d}{2}\right)^2-2\left(1-\frac{d}{2}\right)\right)\left(R^{2n-2}-r^{2n-2}\right)}{4n(n-1)\pi r^{n-1}R^{n-1}} \\ &-\frac{\left(n(n+2)-\left(1-\frac{d}{2}\right)^2-2\left(1-\frac{d}{2}\right)\right)\left(R^{2n+2}-r^{2n+2}\right)}{4n(n+1)\pi r^{n+1}R^{n+1}}\end{aligned}$$

[2]The only difference between $\mathfrak{l}_n(z)$ and $\tilde{\mathfrak{l}}_n(z)$ is the position of r and R.

and

$$\mathfrak{R}_{h_n}(z) = \mathfrak{R}_{\mathfrak{g}_n}(z) + \frac{\left(1-\frac{d}{2}\right)^2}{Rrz^2}\mathfrak{R}_{\mathfrak{f}_n}(z) + \frac{1-\frac{d}{2}}{Rrz}\left(r\mathfrak{R}_{\mathfrak{l}_n}(z) - R\mathfrak{R}_{\tilde{\mathfrak{l}}_n}(z)\right).$$

It is easy to verify that the series on the right hand side of (2.11) is independent of the branch we choose and if $n = d/2-1$, it can be continued analytically onto the negative real axis and the origin. Hence $h_n(z)$ is an entire function and $h_n(0) \neq 0$. If $n \neq d/2-1$, the series can be continued analytically onto the negative real axis. Hence $h_n(z)$ is meromorphic and only has a pole at 0 of the second order. This completes the proof of the case $\nu = n \in \mathbb{N} \setminus \{1\}$.

The proof of other cases are similar and simpler. Here we only point out the differences rather than giving every detail. If $\nu = 1$, when applying the ascending series of Y_n, in addition to (2.6), we also need

$$Y_0(z) = \left(\frac{2\gamma}{\pi}J_0(z) + \frac{2}{\pi}\left(\frac{z}{2}\right)^2 + \cdots\right) + \frac{2}{\pi}\ln\left(\frac{z}{2}\right)J_0(z).$$

See 9.1.13 in [1, P. 360].

If $\nu > 0$ and $\nu \notin \mathbb{N}$, we only need (2.5) to obtain the ascending series of $h_\nu(z)$, although (2.6) is no longer applicable. Combining (1.3) and (1.1) we obtain that

$$\mathfrak{f}_\nu(z) = J_\nu(rz)\frac{J_{-\nu}(Rz)}{\sin(\nu\pi)} - J_\nu(Rz)\frac{J_{-\nu}(rz)}{\sin(\nu\pi)},$$

which is only related to J_ν with $\nu \notin \mathbb{Z}$. The corresponding variants of $\mathfrak{g}_\nu(z)$, $\mathfrak{l}_\nu(z)$ and $\tilde{\mathfrak{l}}_\nu(z)$ follow from (1.3) and the linearity of differentiation. Then by applying (2.5) and the recurrence relation (1.6) to variants of $\mathfrak{f}_\nu(z)$, $\mathfrak{g}_\nu(z)$, $\mathfrak{l}_\nu(z)$ and $\tilde{\mathfrak{l}}_\nu(z)$ yields ascending series of the functions, and hence the series of $h_\nu(z)$ from which we can get the desired results. This completes the proof of the lemma. □

3. Proof of Theorem 1.1

In this section we are going to count the number of zeros of $h_\nu(z)$. In the first half of the proof, we obtain an asymptotics of $h_\nu(z)$ on the circle $|z| = (s+1/2)\pi/(R-r)$, and in the rest we apply the argument principle to $h_\nu(z)$ in the circle to count its zeros.

Proof of Theorem 1.1. By (2.4) and Lemma 2.1, it suffices to count the number of zeros of $h_\nu(z)$.

Let s be an integer with $s > C(\nu^3+1)$ and $C > 0$ to be determined later. By Lemma 2.1 and the argument principle, the number of zeros of $h_\nu(z)$ within the circle σ: $|z| = (s+1/2)\pi/(R-r)$ equals

$$\frac{1}{2\pi i}\int_\sigma \frac{h'_\nu(z)}{h_\nu(z)}\,\mathrm{d}z + 2\delta(\nu), \tag{3.1}$$

where $\delta(\nu)$ is 0 if $\nu = d/2-1$ and 1 otherwise.

We will apply asymptotics of Hankel functions and their derivatives, (1.4) and (1.5) to obtain asymptotics of $\mathfrak{f}_\nu(z)$, $\mathfrak{g}_\nu(z)$, $\mathfrak{l}_\nu(z)$ and $\tilde{\mathfrak{l}}_\nu(z)$ at $|z| = (s+1/2)\pi/(R-r)$, and hence an asymptotic expansion of $h_\nu(z)$. From [7, P. 266–267] we have that if[3] $|z| > A|\nu^2 - 1/4|$ for some constant $A > 0$ and $-\pi/2 \le \arg z \le \pi/2$, then

$$\begin{aligned} H_\nu^{(1,2)}(z) =& \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)}\Bigg(1 \pm i\frac{\mu-1}{8z} - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} \\ &\mp i\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \mathfrak{R}_{H_\nu^{(1,2)}}(z)\Bigg), \end{aligned} \tag{3.2}$$

where the upper (lower) sign applies to the Hankel functions of the first (second) kind, $\mu = 4\nu^2$, and

$$\mathfrak{R}_{H_\nu^{(1,2)}}(z) = O\left((\mu^4+1)|z|^{-4}\right).$$

Moreover, by the recurrence relation (1.6) we have

$$\begin{aligned} H_\nu^{(1,2)\prime}(z) =& \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \left(\frac{\nu}{2}+\frac{1}{4}\right)\pi\right)}\Bigg(\pm i - \frac{\mu+3}{8z} \mp i\frac{(\mu-1)(\mu+15)}{2!(8z)^2} \\ &+ \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3} + \mathfrak{R}_{H_\nu^{(1,2)\prime}}(z)\Bigg), \end{aligned} \tag{3.3}$$

where the upper (lower) sign applies to the Hankel functions of the first (second) kind, and

$$\mathfrak{R}_{H_\nu^{(1,2)\prime}}(z) = O\left((\mu^4+1)|z|^{-4}\right).$$

From the relations (1.4) and (1.5) we have

$$\mathfrak{f}_\nu(z) = \frac{1}{2i}\left(H_\nu^{(2)}(Rz)H_\nu^{(1)}(rz) - H_\nu^{(1)}(Rz)H_\nu^{(2)}(rz)\right).$$

[3]This assumption is satisfied if we take $C > (R-r)A/\pi$.

Then applying the asymptotics (3.2) to all factors in the above formula yields

$$
\begin{aligned}
\mathfrak{f}_\nu(z) = & \frac{-2}{\pi z\sqrt{Rr}}\Bigg(\sin(R-r)z\left(1+\frac{\alpha_2}{(8z)^2}\right) \\
& -\cos(R-r)z\left(\frac{\alpha_1}{8z}-\frac{\alpha_3}{(8z)^3}\right)+\mathfrak{R}_{\mathfrak{f}_\nu}(z)\Bigg),
\end{aligned} \tag{3.4}
$$

where

$$
\alpha_1=(\mu-1)\left(\frac{1}{r}-\frac{1}{R}\right), \quad \alpha_2=\frac{(\mu-1)^2}{Rr}-\frac{(\mu-1)(\mu-9)}{2!}\left(\frac{1}{r^2}+\frac{1}{R^2}\right),
$$

$$
\alpha_3=\frac{(\mu-1)(\mu-9)(\mu-25)}{3!}\left(\frac{1}{r^3}-\frac{1}{R^3}\right)-\frac{(\mu-1)^2(\mu-9)}{2!}\left(\frac{1}{r^2R}-\frac{1}{R^2r}\right)
$$

and

$$
\mathfrak{R}_{\mathfrak{f}_\nu}(z)=O\left((\mu^4+1)|z|^{-4}\right).
$$

From the linearity of differentiation, (1.4), (1.5) and (3.3), the cross-product $\mathfrak{g}_\nu(z)$ has an asymptotic expansion similar to (3.4), but with different coefficients and remainder. Let $\tilde{\alpha}_i$ be its coefficients (corresponding to α_i) with $i=1,2,3$ and $\mathfrak{R}_{\mathfrak{g}_\nu}(z)$ the corresponding remainder. Then

$$
\tilde{\alpha}_1=(\mu+3)\left(\frac{1}{r}-\frac{1}{R}\right), \quad \tilde{\alpha}_2=\frac{(\mu+3)^2}{Rr}-\frac{(\mu-1)(\mu+15)}{2!}\left(\frac{1}{r^2}+\frac{1}{R^2}\right),
$$

$$
\tilde{\alpha}_3=\frac{(\mu-1)(\mu-9)(\mu+35)\left(\frac{1}{r^3}-\frac{1}{R^3}\right)}{3!}-\frac{(\mu-1)(\mu+3)(\mu+15)\left(\frac{1}{r^2R}-\frac{1}{R^2r}\right)}{2!}
$$

and

$$
\mathfrak{R}_{\mathfrak{g}_\nu}(z)=O\left((\mu^4+1)|z|^{-4}\right).
$$

The asymptotics of $\mathfrak{l}_\nu(z)$ and $\tilde{\mathfrak{l}}_\nu(z)$ can be obtained by a similar argument. For $\mathfrak{l}_\nu(z)$ we have

$$
\begin{aligned}
\mathfrak{l}_\nu(z) = & \frac{2}{\pi z\sqrt{Rr}}\Bigg(\cos(R-r)z\left(1+\frac{\delta_2}{(8z)^2}\right) \\
& +\sin(R-r)z\left(\frac{\delta_1}{8z}-\frac{\delta_3}{(8z)^3}\right)+\mathfrak{R}_{\mathfrak{l}_\nu}(z)\Bigg),
\end{aligned} \tag{3.5}
$$

where

$$
\delta_1=\frac{\mu+3}{r}-\frac{\mu-1}{R}, \quad \delta_2=\frac{(\mu-1)(\mu+3)}{Rr}-\frac{(\mu-1)\left(\frac{\mu-9}{R^2}+\frac{\mu+15}{r^2}\right)}{2!},
$$

$$\delta_3 = \frac{(\mu-1)\left(\frac{(\mu-9)(\mu+3)}{R^2 r} - \frac{(\mu-1)(\mu+15)}{Rr^2}\right)}{2!} + \frac{(\mu-1)(\mu-9)\left(\frac{\mu+35}{r^3} - \frac{\mu-25}{R^3}\right)}{3!}$$

and

$$\mathfrak{R}_{\mathfrak{l}_\nu}(z) = O\left((\mu^4+1)|z|^{-4}\right).$$

For $\tilde{\mathfrak{l}}_\nu(z)$, it has an asymptotics similar to (3.5) with different coefficients

$$\tilde{\delta}_1 = \frac{\mu-1}{r} - \frac{\mu+3}{R}, \quad \tilde{\delta}_2 = \frac{(\mu-1)(\mu+3)}{Rr} - \frac{(\mu-1)\left(\frac{\mu-9}{r^2} + \frac{\mu+15}{R^2}\right)}{2!},$$

$$\tilde{\delta}_3 = \frac{(\mu-1)\left(\frac{(\mu-1)(\mu+15)}{R^2 r} - \frac{(\mu-9)(\mu+3)}{r^2 R}\right)}{2!} + \frac{(\mu-1)(\mu-9)\left(\frac{\mu-25}{r^3} - \frac{\mu+35}{R^3}\right)}{3!}$$

and

$$\mathfrak{R}_{\tilde{\mathfrak{l}}_\nu}(z) = O\left((\mu^4+1)|z|^{-4}\right).$$

Then we apply to all factors in $h_\nu(z)$ the asymptotics of $\mathfrak{f}_\nu(z)$, $\mathfrak{g}_\nu(z)$, $\mathfrak{l}_\nu(z)$ and $\tilde{\mathfrak{l}}_\nu(z)$ to obtain

$$\begin{aligned} h_\nu(z) =& \frac{-2}{\sqrt{Rr}\pi z}\Bigg(\sin(R-r)z\left(1+\frac{C_2}{z^2}+\frac{C_4}{z^4}\right) \\ & - \cos(R-r)z\left(\frac{C_1}{z}+\frac{C_3}{z^3}+\frac{C_5}{z^5}\right) + \mathfrak{R}_{h_\nu}(z)\Bigg), \end{aligned}$$

where the coefficients

$$C_1 = \frac{1-\frac{d}{2}}{R} - \frac{1-\frac{d}{2}}{r} + \frac{\tilde{\alpha}_1}{8}, \quad C_2 = \frac{\left(1-\frac{d}{2}\right)^2}{rR} + \frac{\left(1-\frac{d}{2}\right)\tilde{\delta}_1}{8r} - \frac{\left(1-\frac{d}{2}\right)\delta_1}{8R} + \frac{\tilde{\alpha}_2}{8^2},$$

$$C_3 = \frac{\left(1-\frac{d}{2}\right)^2 \alpha_1}{8rR} + \frac{\left(1-\frac{d}{2}\right)\delta_2}{8^2 R} - \frac{\left(1-\frac{d}{2}\right)\tilde{\delta}_2}{8^2 r} - \frac{\tilde{\alpha}_3}{8^3},$$

$$C_4 = \frac{\left(1-\frac{d}{2}\right)^2 \alpha_2}{8^2 rR} + \frac{\left(1-\frac{d}{2}\right)\delta_3}{8^3 R} - \frac{\left(1-\frac{d}{2}\right)\tilde{\delta}_3}{8^3 r}, \quad C_5 = -\frac{\left(1-\frac{d}{2}\right)^2 \alpha_3}{8^3 rR}$$

and the remainder

$$\begin{aligned} \mathfrak{R}_{h_\nu}(z) &= \mathfrak{R}_{\mathfrak{g}_\nu}(z) + \frac{\left(1-\frac{d}{2}\right)^2 \mathfrak{R}_{\mathfrak{f}_\nu}(z)}{rRz^2} - \frac{\left(1-\frac{d}{2}\right)\mathfrak{R}_{\mathfrak{l}_\nu}(z)}{Rz} + \frac{\left(1-\frac{d}{2}\right)\mathfrak{R}_{\tilde{\mathfrak{l}}_\nu}(z)}{rz} \\ &= O\left((\mu^4+1)|z|^{-4}\right). \end{aligned}$$

Notice that the asymptotics of $h_\nu(z)$ above is true for all $|z| = (s+1/2)\pi/(R-r)$ with $|\arg z| \le \pi$ by the evenness of the function.

Now we are ready to count the zeros of $h_\nu(z)$. We first notice that if $|z| = (s+1/2)\pi/(R-r)$, we have

$$\cot(R-r)z = O(1)$$

and

$$\sin^{-1}(R-r)z = O(1).$$

Then a straightforward calculation combined with the above two estimates shows that

(3.6)
$$\frac{1}{2\pi i}\int_\sigma \frac{h_\nu'(z)}{h_\nu(z)}\,\mathrm{d}z = \frac{R-r}{2\pi i}\int_\sigma m_\nu^1(z)+m_\nu^2(z)+m_\nu^3(z)\,\mathrm{d}z - 1 + O\left(\frac{\mu^4+1}{s^3}\right),$$

where

$$m_\nu^1(z) = \cot(R-r)z + \frac{C_1\cot^2(R-r)z}{z} + \frac{C_1}{z},$$

$$m_\nu^2(z) = \frac{\left(\frac{C_1}{R-r}+C_1^2\right)\cot(R-r)z}{z^2} + \frac{C_1^2\cot^3(R-r)z}{z^2},$$

and

$$\begin{aligned} m_\nu^3(z) = &\frac{C_3 - \frac{2C_2}{R-r} - C_1C_2}{z^3} + \frac{\left(C_3 - C_1C_2 + \frac{C_1^2}{R-r} + C_1^3\right)\cot^2(R-r)z}{z^3} \\ &+ \frac{C_1^3\cot^4(R-r)z}{z^3}. \end{aligned}$$

For $m_\nu^1(z)$, the first term $\cot(R-r)z$ has poles $z = m\pi/(R-r)$ of order 1 with $m = 0, \pm1, \pm2, \ldots, \pm s$ within the circle σ. By the residue formula we have

$$\frac{1}{2\pi i}\int_\sigma \cot(R-r)z\,\mathrm{d}z = \sum_{m=-s}^{s}\frac{1}{R-r} = \frac{2s+1}{R-r}.$$

The second term $C_1z^{-1}\cot^2(R-r)z$ has poles $z=0$ of order 3 and $z = m\pi/(R-r)$ of order 2 with $m = \pm1, \pm2, \ldots, \pm s$ within the circle σ. Then

$$\frac{1}{2\pi i}\int_\sigma C_1z^{-1}\cot^2(R-r)z\,\mathrm{d}z = C_1\left(-\frac{2}{3} - \frac{2}{\pi^2}\sum_{m=1}^{s}\frac{1}{m^2}\right).$$

It follows from the equality

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$$

and the estimate $C_1 = O(\mu + 1)$ that

$$\frac{1}{2\pi i}\int_\sigma C_1 z^{-1}\cot^2(R-r)z\,\mathrm{d}z = -C_1 + O\left(\frac{\mu+1}{s}\right).$$

Then from the discussion above we obtain

$$\frac{R-r}{2\pi i}\int_\sigma m_\nu^1(z)\,\mathrm{d}z = 2s + 1 + O\left(\frac{\mu+1}{s}\right). \tag{3.7}$$

Similarly, we have

$$\frac{R-r}{2\pi i}\int_\sigma m_\nu^2(z)\,\mathrm{d}z = O\left(\frac{\mu+1}{s}\right) + O\left(\frac{\mu^2+1}{s^3}\right) \tag{3.8}$$

and

$$\frac{R-r}{2\pi i}\int_\sigma m_\nu^3(z)\,\mathrm{d}z = O\left(\frac{\mu^3+1}{s^3}\right). \tag{3.9}$$

Combining (3.6), (3.7), (3.8) and (3.9) yields

$$\frac{1}{2\pi i}\int_\sigma \frac{h_\nu'(z)}{h_\nu(z)}\,\mathrm{d}z = 2s + O\left(\frac{\mu+1}{s}\right) + O\left(\frac{\mu^4+1}{s^3}\right). \tag{3.10}$$

Notice that $\mu = 4\nu^2$. Hence we can take a sufficiently large constant $C > 0$ such that for any $s \geq C(\nu^3 + 1)$, the right hand side of (3.10) is equal to $2s$ plus a constant with absolute value less than 1. On the other hand, the integral equals an integer by the argument principle, which implies that

$$\frac{1}{2\pi i}\int_\sigma \frac{h_\nu'(z)}{h_\nu(z)}\,\mathrm{d}z = 2s. \tag{3.11}$$

Then the desired results follow from (3.1) and (3.11). This completes the proof of Theorem 1.1. □

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